Math 254B Lecture 18 Notes

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1 Fractals and Fractal Dimension

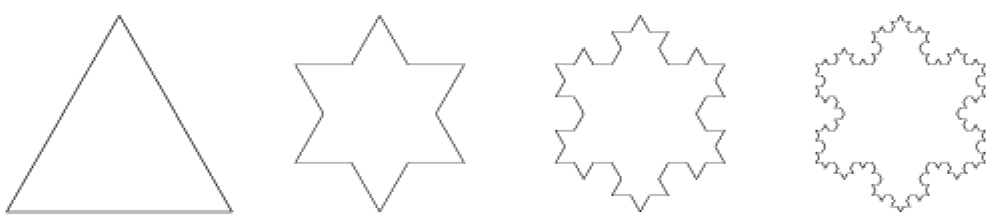
1.1 Examples of fractals

A fractal[1] is some kind of metric space with no smooth structure. Here are some examples of fractals.[2]

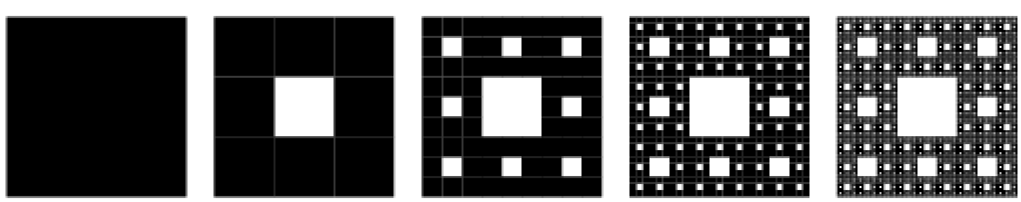
Example 1.1. The **middle-α Cantor set** is a fractal:

Alternatively, if we have $(\alpha_1, \alpha_2, \dots)$, we can form the **middle-α_j Cantor set** by taking away the middle α_j from each remaining interval at step j.

Example 1.2. The von Koch curve is another fractal:

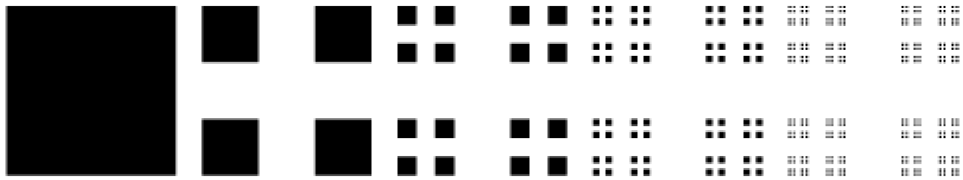


Example 1.3. The **Sierpiński carpet** is a fractal:



[1] People have generally agreed that we should not have a strict definition of fractal.
[2] Images taken from Wolfram mathworld.

Example 1.4. Here is a fractal called **Cantor dust**:



These are all special non-smooth spaces; all of these are generated in some recursive way.

1.2 Box-counting dimension

How do we talk about sizes of fractals?

Definition 1.1. Let (X, ρ) be a metric space, and let $A \subseteq X$. Let (X, ρ) The **upper box-counting dimension** of A is

$$\overline{\dim}_B(A) := \limsup_{\delta \to 0} \frac{\log(\operatorname{cov}_\delta(A))}{\log(\delta^{-1})}.$$

The **lower box-counting dimension** of A is

$$\underline{\dim}_B(A) := \liminf_{\delta \to 0} \frac{\log(\operatorname{cov}_\delta(A))}{\log(\delta^{-1})}.$$

If these are equal, we call them the **box-counting dimension** $\dim_B(A)$.

Lemma 1.1. *We get the same value if we use*

1. *Closed balls instead of open,*
2. $\operatorname{cov}_\delta$*,*
3. $\operatorname{pack}_\delta$*,*
4. *If* $X = \mathbb{R}^n$*, use cubes and let* δ *be the diameter of the cubes,*
5. *In* $X = \mathbb{R}^n$*, we also have*

$$\underline{\dim}_B(A) = \liminf_{n \to \infty} \frac{\log(N(A, b^n))}{n \log(b)},$$

where $N(A, b^n)$ *is the number of* b*-adic cubes (with vertices in* $(b^{-n}\mathbb{Z})^d$*) which intersect* A*.*

Proof. The real reason is that

$$\overline{\dim}_B(A) = \limsup_{j\to\infty} \frac{\log(\mathrm{cov}_{\delta_j}(A))}{\log(\delta_j^{-1})}$$

whenever $\delta_j \downarrow 0$, provided $\sup_j \delta_j/\delta_{j+1} = C < \infty$. If $\delta_{j+1} \le \delta < \delta_j$, then

$$\frac{\log(\mathrm{cov}_\delta(A))}{\log(\delta^{-1})} \ge \frac{\log(\mathrm{cov}_\delta(A))}{\log(\delta^{-1})} \cdot \frac{\log(\delta^{-1})}{\log(\delta^{-1}) + \log(C)}.$$

The same holds for the lower box-counting dimension.

(1) $\iff$ (4): observe that

$$3^{-d} N(A, b^{-n}) \le \mathrm{cov}_{b^{-n}}(A) \le 3^d N(A, b^n).$$

□

Example 1.5. Let C_α be the middle-α Cantor set. For every i, we can cover C_α using 2^i intervals of length $\left(\frac{1-\alpha}{2}\right)^i$. So we get

$$\overline{\dim}_B(C_\alpha) \le \frac{\log(2)}{\log(2/(1-\alpha))}.$$

You can show that this is actually the box-covering dimension.

For $C_{\langle \alpha_j \rangle}$, we will be covering with intervals of length $(1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_i)/2^i$. So we get

$$\overline{\dim}_B(C_{\langle \alpha_j \rangle}) = \limsup_{i\to\infty} \frac{\log(2)}{\frac{1}{i}\sum_{j=1}^i (\log(2) - \log(1-\alpha_j)},$$

$$\underline{\dim}_B(C_{\langle \alpha_j \rangle}) = \liminf_{i\to\infty} \frac{\log(2)}{\frac{1}{i}\sum_{j=1}^i (\log(2) - \log(1-\alpha_j)}.$$

Definition 1.2. If $f : (X, \rho_X) \to (Y, \rho_Y)$ and $0 < \alpha \le 1$, then f is **α-Hölder** if there is a $C < \infty$ such that

$$\rho_Y(f(x), f(y)) \le C\rho(x, y)^\alpha.$$

Lemma 1.2. *If f is α-Hölder, then*

$$\overline{\dim}_B(f(A)) \le \frac{\overline{\dim}_B(A)}{\alpha},$$

$$\underline{\dim}_B(f(A)) \le \frac{\underline{\dim}_B(A)}{\alpha}.$$

Proof. If we cover A by $B(x_1, \delta), \ldots, B(x_n, \delta)$¡ the n$f(A) \subseteq \bigcup_i B(f(x_i), C\delta^\alpha)$. □

Lemma 1.3. $\overline{\dim}_B(A) = \overline{\dim}_b(\overline{A})$.

Corollary 1.1. $\dim_B(\mathbb{Q} \cap [0, 1]) = 1$.

This does not quite agree with what we think should be small, so we introduce a different notion.

1.3 Hausdorff dimension

Definition 1.3. Let $A \subseteq X$, and $0 \leq \alpha < \infty$. Then the **α-dimensional Hausdorff content**[3] is

$$\mathcal{H}^\alpha_\infty(A) := \inf \left\{ \sum_i \operatorname{diam}(E_i)^\alpha : A \subseteq \bigcup_i E_i \right\}.$$

Lemma 1.4. *If $0 < \alpha < \beta$ and $\mathcal{H}^\alpha_\infty(A) = 0$, then $\mathcal{H}^\beta_\infty(A) = 0$.*

Proof. Assume that for every $\varepsilon > 0$, there are E_i such that $A \subseteq \bigcup_i E_i$ and $\sum_i \operatorname{diam}(E_i)^| alpha) < \varepsilon$. Then $\operatorname{diam}(E_i) \leq \varepsilon^{1/\alpha}$ for all i. Then

$$\sum_i \operatorname{diam}(E_i)^{(\beta-\alpha)+\alpha} \leq \varepsilon^{(\beta-\alpha)/\alpha} \sum_i \operatorname{diam}(E_i)^\alpha \leq \varepsilon^{(\beta-\alpha)/\alpha+1}. \qquad \square$$

Definition 1.4. The **Hausdorff dimension** is $\dim(A) = \dim_H(A) = \inf\{\alpha > 0 : \mathcal{H}^\alpha_\alpha(A)\}$.

Proposition 1.1. *We get the same notion of dimension using open/closed balls, b-adic cubes, etc.*

Lemma 1.5. $\dim(A) \leq \underline{\dim}_B(A)$.

Proof. For any $\alpha > \underline{\dim}_B(A)$. there exist $\delta_1 > \delta_2 > \cdots \to 0$ such that $\operatorname{cov}_{\delta_j}(A) \leq C\delta_j^{-\alpha}$. If $B(x_1, \delta_j), \dots, B(x_N, \delta_j)$ are these balls, we get for any $\alpha' > \alpha$ that

$$B^{\alpha'}_\infty(A) \leq \sum_{n=1}^N (2\delta_j)^{\alpha'} \leq 2^{\alpha'} C \delta_j^{\alpha'} \delta_j^{-\alpha} = O(1)\delta_j^{\alpha'-\alpha} \to 0. \qquad \square$$

So we have that

$$\dim \leq \underline{\dim}_B \leq \overline{\dim}_B.$$

Definition 1.5. A is **exact dimensional** if these are all equal.

Definition 1.6. Define $\mathcal{H}^\alpha_\delta(A) := \inf \{\sum_i \operatorname{diam}(E_i)^\alpha : A \subseteq \bigcup_i E_i, \operatorname{diam}(E_i) \leq \delta\}$. The **$\alpha$-dimensional Hausdorff measure** is

$$m_\alpha(A) := \lim_{\delta \to 0} \mathcal{H}^\alpha_\delta(A).$$

[3] One way to think about this is as an infinite analogue of cov′